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Christoffel Formula and Geodesic Motion in Hyperspin Manifolds

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A hyperspin manifold S_N constructed from N-component hyperspinors is an alternative to Riemannian manifolds R^n for Kaluza-Klein-type theories of higher dimensions. Hyperspin manifolds possess a fundamental chronometric tensor with N n-valued indices, where always $n = N^2$. Some concepts of Riemannian geometry therefore have to be extended. A hyper-Christoffel formula is presented that expresses the connection in terms of the chronometric, assuming the chronometric is covariantly constant and the connection is torsion-free. Thus, the chronometric can be used as sole dynamical variable. Extremals and self-parallel curves, which coincide in Riemannian manifolds, in general differ in hyperspin manifolds, but coincide again for nonnull curves.

1. INTRODUCTION

Spin manifolds based on Weyl two-component spinors are introduced by Infeld and van der Waerden (1933), Taub and Givens (1955), and Penrose (1960) to describe four-dimensional time-spaces. Hyperspin manifolds S_N built from N-component hyperspinors of $SL(N, C)$ were suggested by Finkelstein (1986) to support a Kaluza-Klein theory of gauge potentials. A hyperspin manifold gives rise to a causal time-space of dimension $n = N^2$, which possesses a symmetric chronometric form $g_{a_1 \dots a_N}$ with N n-valued indices. Proper time $d\tau$ is given by

$$d\tau^N = g_{b_1 \dots b_N} dt^{b_1} \cdots dt^{b_N} = N \det(dt^{A\dot{A}}) \tag{1}$$

where $t^{A\dot{A}}$ are the sesquispinors, $N \times N$ Hermitian matrices, of the spin manifold, related to the time-space vectors via the spin vector, a local isomorphism (Finkelstein et al., 1986).

The geometry of S_N is no longer Riemannian. In this paper I present a Christoffel formula for n-dimensional hyperspin manifolds and show that

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there are two kinds of geodesics present, which are equivalent for special nonnull curves.

2. HYPER-CHRISTOFFEL FORMULA

In Einstein's gravitational theory, the fundamental chronometric form is covariantly constant (isometric) with respect to the vector connection: $D_a g_{bc} = 0$. This leads to the well-known Christoffel formula in a holonomic or coordinate basis.

In Riemannian geometry only the Christoffel formula permits us to use the chronometric as sole dynamical variable in the variation of the Lagrangian. For the Hilbert action the Christoffel formula can be obtained via the Palatini (1919) method. This is not possible for higher derivative theories, however. A hyper-Christoffel formula therefore gives us the possibility of using the chronometric alone. This leads to simpler equations of motion.

Under the assumption that the chronometric is also covariantly constant in hyperspin manifolds, the derivatives of the chronometric will determine the torsion-free, symmetric vector connection Γ, resulting in a hyper-Christoffel formula.

Thus

$$D_a g_{b_1 \ldots b_N} = 0$$

implies

$$\partial_a g_{b_1 \ldots b_N} - \Gamma^s_{ab_1} g_{sb_2 \ldots b_N} - \cdots - \Gamma^s_{ab_N} g_{sb_1 \ldots b_{N-1}} = 0 \tag{2}$$

Define a *hyper-Christoffel symbol of the first kind*, symmetric in both sets of indices:

$$g_{b_2 \ldots b_N s} \Gamma^s_{ab_1} =: [ab_1, b_2 \ldots b_N] \tag{3}$$

Assume a solution of the form

$$[ab_1, b_2 \ldots b_N] = A(\partial_a g_{b_1 \ldots b_N} + \partial_{b_1} g_{ab_2 \ldots b_N}) + B(\partial_{b_2} g_{\ldots} + \cdots + \partial_{b_N} g_{\ldots}) \tag{4}$$

This gives

$$[ab_1, b_2 \ldots b_N] = N^{-1}(\partial_a g_{b_1 \ldots b_N} + \partial_{b_1} g_{ab_2 \ldots b_N}) - [N(N-1)]^{-1}(\partial_{b_2} g_{\ldots} + \cdots + \partial_{b_N} g_{\ldots}) \tag{5}$$

Using the *dual chronometric*, defined by

$$g^{b_2 \ldots b_N s} g_{tb_2 \ldots b_N} = \delta^s_t \tag{6}$$

one obtains the *hyper-Christoffel symbols of the second kind*:

$$\Gamma^t_{ab_1} = g^{b_2 \ldots b_N t}[ab_1, b_2 \ldots b_N] \tag{7}$$

The solution is necessarily unique, because (2) forms $N^4(N^2+1)\cdots(N^2+N-1)/N!$ equations for only $N^4(N^2+1)/2\,\Gamma$'s. Only in the case $N=2$ is the number of equations equal to the number of Γ's. Otherwise, the Christoffel symbols are overdetermined by (2). In the case $N=2$, (5) and (7) of course reduce to Christoffel's familiar formula.

3. GEODESICS

I call a parametrized curve $t^a(\lambda)$ a *self-parallel* curve if its tangent vector $v := dt^a/d\lambda$ is parallel-transported along itself:

$$v^a D_a v^b = 0 \tag{8}$$

I call a curve *extremal* if its proper time $d\tau$ is an extremum, i.e.,

$$\delta \int d\tau = \int |dt^a/d\lambda|^{1/N}\, d\lambda = 0 \tag{9}$$

I call a curve *geodesic* if it is both extremal and self-parallel.

To simplify the calculation of (9), the variation is applied only to $d\tau^N$, which is permissible for a nonnull arc-length parameter, and carried out in a geodesic coordinate system where the Γ's are all zero. Using the isometry of g also, one obtains

$$g_{b_2\ldots b_N s} v^{b_2} \cdots v^{b_{N-1}} v^a D_a v^{b_N} = 0 \tag{10}$$

One sees immediately that the self-parallel equation is a sufficient condition for the extremal one. Whether it is also a necessary one depends on the invertibility of the *reduced chronometric* $g_{b_2\ldots b_N s} v^{b_2} \cdots v^{b_{N-1}}$, which is a second-rank tensor.

A chart is called *determinantal* if in it the spin map from the space of sesquispinors to the tangent bundle of time-space is a simple delta function. In such a chart the invertibility can be proven for special cases. The chronometric is then given by

$$g_{a_1\ldots a_N} = \varepsilon_{AB\ldots N}\varepsilon_{\dot{A}\dot{B}\ldots\dot{N}}/(N-1)! \tag{11}$$

where the N indices $A, B, \ldots, N$ range from $1, \ldots, N$.

From (1) one sees that the determinant of a nonnull vector is nonzero. It can be diagonalized by an $SL(N, C)$ transformation to a unique sequence (up to order) of ∓ 1 (normal form). A future timelike vector has $\delta^{A\dot{A}}$ as its normal form.

Lemma. For future-timelike vectors, the reduced chronometric is invertible.

Proof. Write $\varepsilon_{AB\ldots N}\varepsilon_{\dot{A}\dot{B}\ldots\dot{N}}$ in its delta-function representation,

$$\varepsilon_{AB\ldots N}\varepsilon_{\dot{A}\dot{B}\ldots\dot{N}}=\det\begin{bmatrix}\delta_{A\dot{A}} & \cdots\cdots\cdots & \delta_{A\dot{N}}\\ \vdots & & \vdots\\ \delta_{N\dot{A}} & \cdots\cdots\cdots & \delta_{N\dot{N}}\end{bmatrix} \tag{12}$$

By (11) the reduced chronometric then has the form

$$\varepsilon_{AB\ldots N}\varepsilon_{\dot{A}\dot{B}\ldots\dot{N}}\delta^{C\dot{C}}\cdots\delta^{N\dot{N}}$$

which is proportional to $\delta_{A\dot{A}}\delta_{B\dot{B}}-\delta_{A\dot{B}}\delta_{B\dot{A}}$. Acting upon any nonzero matrix, the resulting matrix is always nonzero. ■

The normal form of spacelike vectors is a mixed sequence of ∓ 1. The spacelike vectors therefore cannot be represented by a simple delta function, which makes the computations by use of (12) rather lengthy. The calculations done up to $N=4$ show the same result as before and it seems likely that the reduced chronometric is invertible for any nonnull vector. This suggests that extremal and self-parallel curves are equivalent for nonnull curves in hyperspin manifolds as well as in Riemannian manifolds.

For a nullvector, i.e., a sesquispinor having at least one zero eigenvalue, one can show by using (12) that the reduced chronometric is not invertible. Hence, there exist extremal curves that are not self-parallel. These curves might be important motions for free particles or fields.

4. RESULTS

In isometric, torsion-free hyperspin manifolds the connection can be uniquely expressed in terms of derivatives of the chronometric. This leaves the chronometric as the single dynamical variable and source of gauge fields. Geodesics are defined in hyperspin manifolds analogously to Riemannian manifolds. All self-parallel curves and all nonnull extremals are geodesics, but only special null extremals are.

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